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A general characterization of interim efficient mechanisms for independent linear environments

John O. Ledyard[a], Thomas R. Palfrey[b,*]

[a] *Division of Humanities and Social Sciences, California Institute of Technology, Mail Code 228-77, Pasadena, California, USA 91125*

[b] *Departments of Politics and Economics, Princeton University, Princeton, NJ, USA 08544*

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Abstract

We consider the class of Bayesian environments with independent types, and utility functions which are both quasi-linear in a private good and linear in a one-dimensional private-value type parameter. We call these *independent linear environments*. For these environments, we fully characterize interim efficient allocation rules which satisfy interim incentive compatibility and interim individual rationality constraints. We also prove that they correspond to decision rules based on virtual surplus maximization, together with the appropriate incentive taxes. We illustrate these techniques with applications to auction design and public good provision.
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1. Introduction

Many papers have now been written on optimal mechanism design for Bayesian environments.[1] While a variety of technical approaches have been taken, most of these papers share a common mathematical structure, but this common structure is not transparent, as these techniques are scattered across a number of articles, each of which focuses on a specific application or feature

* Corresponding author. Fax: +1 609 258 8854.
E-mail addresses: jledyard@hss.caltech.edu (J.O. Ledyard), tpalfrey@princeton.edu (T.R. Palfrey).

[1] See, for example, Cornelli [4], Coughlan [5], Cramton and Palfrey [7,8], Crémer et al. [9], d'Aspremont and Gérard-Varet [1], Dudek et al. [11], Gresik [12], Laffont and Maskin [19,20], Ledyard and Palfrey [23–26], Mailath and Postlewaite [29], Makowski and Mezzetti [30], Myerson and Satterthwaite [33], Myerson [32], Wilson [41,42].

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of the general problem. Here, we exploit that common structure to give a full characterization of interim efficient allocation rules for what we call *linear independent environments*. These environments have quasi-linear utility, additivity in taxes in the feasibility constraints, and linearity of utilities in a one-dimensional independent private-value type. The general model embodies both public good problems and private good problems in a single framework. We also prove that these solutions correspond to decision rules based on a virtual cost-benefit criterion, together with the appropriate incentive taxes.

As is standard, we use the revelation principle to characterize efficient allocation rules by restricting attention to direct revelation mechanisms. We use the separation result of d'Aspremont and Gérard-Varet [1] which allows the separate computation of feasible incentive taxes. We will use an insight of Myerson and Satterthwaite [33] which reduces individual rationality constraints to a single constraint that does not involve the incentive taxes. The technical approach is closest to the original Mirrlees [31] analysis of optimal taxation for income redistribution, and Wilson's [42] later study of ex ante optimal trading procedures.

In contrast to the above papers, this paper is concerned with interim efficient allocation rules, using a concept first introduced by Holmstrom and Myerson [17]. An allocation rule is interim efficient if there exists no other allocation rule that makes no type of any agent worse off and makes some types of some agents better off. It is the natural generalization of Pareto optimality to Bayesian environments where agents have private information. There are only a handful of papers that explore the properties of interim efficient allocation rules, and these are limited to a few applications. [2]

The next section presents the basic notation and the model. Section 3 presents the characterization results and proofs. Section 4 shows how the characterization is simplified in the regular case and Section 5 illustrates this approach with several applications to both public and private goods environments. We make some concluding remarks in Section 6.

2. The model

There are N individual agents. An *outcome* consists of a social allocation and a profile of taxes. A *social allocation* is an M-vector, denoted $x = (x^1, \dots, x^M)$ which is an element of a feasible set $X \subseteq R^M$ for some $M > 0$. The *cost* of the social allocation is given by $C(x)$, and $a = (a^1, \dots, a^N) \in R^N$ is a *profile of taxes* for the agents, which must collectively be sufficient to cover the cost of x. We denote the set of feasible profiles of taxes, given an allocation x, by $\mathbf{A}(x) = \{a \in R^N \mid \sum_{i=1}^N a^i \geqslant C(x)\}$. Formally, a *feasible outcome* is a pair $(x, a) \in Z$ where Z is the subset of $X \times R^N$ such that $a \in \mathbf{A}(x)$ for all $x \in X$.

Each player has a type, t^i. We assume that each individual knows his own type and does not know the types of the other individuals. We assume that the types are *independently distributed*, with the (common knowledge) cdf of i's type denoted $F_i(\cdot)$ and the support of F_i is $T^i = [\underline{t}^i, \bar{t}^i] \subseteq R$. We assume F_i has a continuous strictly positive density on T_i. Note that $\underline{t}^i < 0$ is allowed. The von Neumann Morgenstern utility function for type t^i of agent i for an allocation (x, a) is assumed to take the form [3] $V^i(x, a, t^i) = t^i q^i(x) - a^i$.

[2] See Gresik [12] and Wilson [41] for applications to bilateral trade, particularly double auctions. See Coughlan [5], Laussel and Palfrey [22] and Ledyard and Palfrey [23–26] for applications to public good mechanisms. Perez-Nievas [35] investigates the interim efficiency of Groves mechanisms.

[3] In many applications, $q^i(x)$ is the quantity consumed by agent i in the social allocation x. However, this is just one of several possible interpretations of q.

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An *allocation rule* is a mapping from $T = T^1 \times \cdots \times T^N$ into Z. A *mechanism* is a game form consisting of a message set for each agent and an outcome function that maps message profiles into probability distributions over the set of feasible allocations. A *direct mechanism* is a mechanism in which the message set for each agent is simply T^i. By the revelation principle, any allocation rule that results from equilibrium in any mechanism is also an equilibrium allocation rule of an incentive compatible, direct mechanism. Therefore, the rest of the paper only considers direct mechanisms.

A strategy for i in a direct mechanism is a mapping $\sigma^i : T^i \to T^i$: that is, a decision rule that specifies a reported type for each possible type. We denote a feasible direct mechanism simply as a function, $\eta : T \to Z$. We denote the social allocation component of η at type profile t by $x(t)$ and the tax profile by $a(t)$. We will refer to the pair $(q^i(x(\cdot)), a^i(\cdot))$ as i's allocation under η.

Besides resource feasibility, the two restrictions on η considered in this paper are incentive compatibility and individual rationality. Incentive compatibility requires that it is a Bayesian equilibrium of η for all agents to adopt a strategy of truthfully reporting their type. Given a strategy profile $\sigma^i : T^i \to T^i$ and mechanism, η, denote by $\widehat{U}^i(\eta, t^i, s^i)$ the interim utility to type t^i of agent i, if he reports type s^i, assuming all other agents truthfully report their type. That is

$$\widehat{U}^i(\eta, t^i, s^i) = \int_T [t^i q^i[x(s^i, t^{-i})] - a^i(s^i, t^{-i})]\, dF(t \mid t^i).$$

And, denote $U^i(\eta, t^i) \equiv \widehat{U}^i(\eta, t^i, t^i)$.

Definition 1. A direct mechanism η is (interim) *incentive compatible* if and only if $U^i(\eta, t^i) \geqslant \widehat{U}^i\ (\eta, t^i, s^i)$ for all i, t^i, s^i.

We also require allocation rules η to satisfy an interim individual rationality constraint. This means each type of each agent must be at least as well off, at the interim stage, by participating, as they would be by not participating, assuming truthful reporting by all agents. We assume the interim expected utility of not participating in the mechanism does not depend on the mechanism, but can depend on type. We denote this non-participation value by $U^{0i}(t^i)$. [4]

Definition 2. A direct mechanism η satisfies (interim) *individual rationality* if and only if $U^i(\eta, t^i) \geqslant U^{0i}(t^i)$ for all i, t^i.

Definition 3. A direct mechanism η is *interim efficient* iff (a) η is feasible, (b) η is (interim) incentive compatible and (c) η satisfies (interim) individual rationality and $\nexists\ \widehat{\eta}$ such that $\widehat{\eta}$ is feasible, $\widehat{\eta}$ is (interim) incentive compatible, and $\widehat{\eta}$ satisfies (interim) individual rationality, such that $U^i(\widehat{\eta}, t^i) \geqslant U^i(\eta, t^i)$ for all i, t^i, and $U^i(\widehat{\eta}, t^i) > U^i(\eta, t^i)$ for some i and for all $t^i \in \widetilde{T}^i \subset T^i$, where $\widetilde{T}^i$ has strictly positive measure relative to T^i.

The following well-known result [5] is stated below, without proof.

Theorem 1. *A direct mechanism η is an interim efficient mechanism iff $\exists$ a set of interim welfare weights, $\lambda = \left\{\lambda^i : T^i \to R_+\right\}_{i=1}^N$ with $\int_{\underline{t}^i}^{\bar{t}^i} \lambda^i(t^i)\, dF^i(t^i) > 0$ for some i, such that*

[4] In this formulation, $U^{0i}(t^i)$ is taken to be exogenous and possibly type dependent. In many applications it is assumed that $U^{0i}(t^i) = 0$ for all types, but in general it can depend on type.

[5] See Holmstrom and Myerson [17].

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η maximizes $\sum_{i=1}^{N}\int_{\underline{t}^i}^{\bar{t}^i}\lambda^i(t^i)U^i(\eta(t),t^i)\,dF^i(t^i)$ subject to (a) *η is feasible,* (b) *η is* (*interim*) *incentive compatible and* (c) *η satisfies the* (*interim*) *individual rationality constraint.*

We now proceed to characterize that set of interim efficient mechanisms.

3. The characterization

The characterization of interim efficient mechanisms is broken down into two parts. First, the constraints in Lemma 1 (incentive compatibility, resource feasibility, and interim individual rationality, or voluntary participation) are characterized. The constraints for incentive compatibility correspond to first and second order conditions of an individual optimization problem, using standard arguments. The constraints for resource feasibility and interim individual rationality are simple inequalities, but take a more convenient form when one substitutes in the incentive constraints. Obtaining this more convenient form of the interim individual rationality constraint is not completely standard and requires some additional notation, so we explain it in more detail. Following this, we illustrate it with the bilateral bargaining problem studied by Chatterjee and Samuelson [3] and Myerson and Satterthwaite [33].

The second step in the characterization involves a general solution to the maximization problem posed in Lemma 1, with the constraints rewritten as described above. We show how this problem simplifies in the so-called "regular case" where the second order incentive compatibility condition is not a binding constraint.

3.1. Constraints

3.1.1. The incentive compatibility constraint

Incentive compatibility is satisfied if and only if $U^i(\eta,t^i)\geqslant\widehat{U}^i(\eta,t^i,\sigma^i)$ for all i, t^i, σ^i. When preferences are linear in type and η is twice differentiable, this can be rewritten in terms of two simple conditions [37]. First, an envelope condition specifies that the total derivative of the interim utility for *i* with respect to type when players adopt truthful strategies is equal to the partial derivative with respect to type (i.e., fixing the reports of all agents). Second, the second derivative interim utility to *i* with respect to t^i under truthful reporting is positive, so U^i convex in *i*'s type.

In linear independent environments, this characterization of incentive compatibility can be expressed very simply and generally in terms of constraints on *reduced form allocations*; that is, the expected value of that type's allocation under the mechanism, when all agents report truthfully. The reduced form social allocation of type t^i is denoted is $Q^i(t^i)\equiv\int_T q^i[x(t)]\,dF(t\mid t^i)$, and type t^i's reduced form tax is denoted by $A^i(t^i)\equiv\int_T a^i(t)\,dF(t\mid t^i)$. In the differentiable case, the envelope condition reduces to $t^i\frac{\partial Q^i}{\partial t^i}=\frac{\partial A^i}{\partial t^i}$ and the convexity condition is $\frac{\partial Q^i}{\partial t^i}\geqslant 0\ \forall i$, and $t^i\in T^i$. This well known result is stated slightly more generally below, and allows for mechanisms that may not be everywhere differentiable.

Lemma 1. *A direct mechanism η is incentive compatible iff, for all i, $t^i\in T^i$,*

$$U^i(\eta,t^i)=U^i(\eta,\underline{t}^i)+\int_{\underline{t}^i}^{t^i}Q^i(s)\,ds \qquad \text{(IC1)}$$

and

$$Q^i(t^i) \text{ is non-decreasing in } t^i. \tag{IC2}$$

3.1.2. The resource feasibility constraint

Resource feasibility, or (weak) budget balancing, requires that, for every realization of types, enough taxes are raised to pay the cost. That is, $\sum_{i=1}^N a^i(t) \geqslant C(x(t))\ \forall t$. Like incentive compatibility, this constraint is easier to work with by transforming it into reduced form, and substituting the incentive constraint. With this in mind we define the expected budget surplus of an incentive compatible allocation rule (summed over all agents).

Definition 4. Given an allocation rule x the *expected budget surplus* can be rewritten as

$$\begin{aligned} S(x) \equiv \sum_{i=1}^N &\left[\int_{\underline{t}^i}^{\bar{t}^i} \left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)} \right) Q^i(t^i)\, dF^i(t^i) - \underline{t}^i Q^i(\underline{t}^i) + A^i(\underline{t}^i) \right] \\ &- \int_T C(x(t))\, dF(t). \end{aligned}$$

To see this is indeed the expected budget surplus, observe that substitution of the incentive compatibility constraint (IC1) into the definition of expected taxes for i gives

$$\int_{\underline{t}^i}^{\bar{t}^i} A^i(t^i)\, dF^i(t^i) = \int_T \left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)} \right) q^i(x(t))\, dF(t) - U^i(\underline{t}^i).$$

Hence, $S(x)$ is the ex ante budget surplus, given incentive compatible taxation. The next step involves the separation of the transfer problem (the choice of a) from the allocation problem (the choice of x), which is a well-known result.[6] We include a proof for completeness.

Lemma 2. *Let $x : T \to X$ be an allocation rule such that Q^i is a non-decreasing function of t_i for all i. If $\{A^{0i}\}_{i=1}^N$ is a collection of N constants, then $\exists\, a : T \to R^N$ such that (x, a) is incentive compatible and feasible and $A^{0i} = A^i(\underline{t}^i)$ for all i, if and only if $S(x) \geqslant 0$.*

Proof. See Appendix. □

The inequality in the statement of the lemma requires that given incentive compatible taxation, ex ante expected taxes are greater than or equal to ex ante expected costs. In other words, it is only the ex ante budget balance constraint that is binding.

3.1.3. The individual rationality constraint

Individual rationality, or voluntary participation, is satisfied if and only if $U^i(\eta, t^i) \geqslant U^{0i}(t^i)$ for all i, t^i. The key is to obtain an equivalent *usable* version of this constraint in terms of a *single* inequality constraint. This is done by combining it with incentive constraints.

Because we are interested in a range of applications from public goods to bargaining to auctions, we need expressions which will accommodate both buyers and sellers. Individual rationality requires an agent's *net utility* given incentive taxes to be non-negative for all of that

[6] See d'Aspremont and Gérard-Varet [1].

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agent's types:

$$U^i(\eta, t^i) - U^{0i}(t^i) = t^i Q^i(t^i) - A^{0i} - \int_{\underline{t}^i}^{t^i} s\, dQ^i(s) - U^{0i}(t^i) \geqslant 0\ \forall t^i \in T^i.$$

That is, it requires

$$\begin{aligned} 0 &\leqslant \min_{t^i}\left[t^i Q^i(t^i) - A^{0i} - \int_{\underline{t}^i}^{t^i} s\, dQ^i(s) - U^{0i}(t^i)\right] \\ &\Leftrightarrow \\ 0 &\leqslant \min_{t^i}\left[\underline{t}^i Q^i(\underline{t}^i) - A^{0i} + \int_{\underline{t}^i}^{t^i} Q^i(s)\, ds - U^{0i}(t^i)\right] \\ &\Leftrightarrow \\ A^{0i} &\leqslant \underline{t}^i Q^i(\underline{t}^i) + \min_{t^i}\left[\int_{\underline{t}^i}^{t^i} Q^i(s)\, ds - U^{0i}(t^i)\right]. \end{aligned}$$

Notice that for buyers with $U^{0i}(t^i) = 0$ for all t^i as in the standard auction or public goods applications, since $Q^i \geqslant 0$, this reduces to $U^i(\underline{t}^i) = \underline{t}^i Q^i(\underline{t}^i) - A^{0i} \geqslant 0$. But for sellers, for whom $U^{0i}(t^i) = t^i$, this requires $U^i(\underline{t}^i) + \int_{\underline{t}^i}^{\bar{t}^i} Q^i(s)\, ds - \bar{t}^i \geqslant 0$. There may also be applications for which voluntary participation binds in the interior, so the argmin is neither $\bar{t}^i$ or $\underline{t}^i$ [6].

We next combine individual rationality with feasibility to get a useful result for later.

Lemma 3. *If $x : T \to X$ satisfies $\frac{\partial Q^i}{\partial t^i} \geqslant 0$, then there exists $a : T \to R^N$ such that (x, a) is incentive compatible, feasible, and satisfies individual rationality if and only if*

$$\sum_{i=1}^{N}\left[\int_{\underline{t}^i}^{\bar{t}^i}\left(t^i - \frac{1 - F^i(t^i)}{f^i(t^i)}\right) Q^i(t^i)\, dF^i(t^i) + \min_{t^i}\left[\int_{\underline{t}^i}^{t^i} Q^i(s)\, ds - U^{0i}(t^i)\right]\right] - \int_T C(x(t))\, dF(t) \geqslant 0. \quad (1)$$

Proof. See the Appendix. □

To see how easily existing results follow from Lemma 3, we look at one early application as an illustrative example.

Example. Myerson and Satterthwaite [33], consider a bargaining problem with two agents, a buyer, B, and a seller, S, and $C(x) = 0$. Each trader can guarantee himself the no trade outcome. B has no endowment, but S has the option to keep the object and receive $U^{0i}(t^i) = t^i$. Therefore, as we found above,

$$\min_{t^i}\left[\int_{\underline{t}^B}^{t^B} Q^B(s)\, ds - U^{0B}(t^B)\right] = 0 \quad \text{for } B$$

and

$$\min_{t^S}\left[\int_{\underline{t}^S}^{t^S} Q^S(s)\, ds - U^{0S}(t^S)\right] = \bar{t}^S - \int_{\underline{t}^S}^{\bar{t}^S} Q^S(s)\, ds \quad \text{for } S.$$

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Using these two identities and constraint (1) yields the inequality for the IR constraint:

$$\int_{\underline{t}^B}^{\bar{t}^B} \left(t^B - \frac{1-F^B(t^B)}{f^B(t^B)}\right) Q^B(t^B)\, dF^B(t^B) + \int_{\underline{t}^S}^{\bar{t}^S} \left(t^S + \frac{F^S(t^S)}{f^S(t^S)}\right) Q^S(t^S)\, dF^S(t^S) \geqslant 0.$$

From Lemma 3, one can see that this is easily extended to an arbitrary number of buyers and sellers. Denoting the *set* of buyers by B and the *set* of sellers by S, we the participation constraint becomes

$$\sum_{i\in B} \int_{\underline{t}^i}^{\bar{t}^i} \left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)}\right) Q^i(t^i)\, dF^i(t^i) + \sum_{j\in S} \int_{\underline{t}^j}^{\bar{t}^j} \left(t^j + \frac{F^j(t^j)}{f^j(t^j)}\right) Q^j(t^j)\, dF^j(t^j) \geqslant 0.$$

3.2. *Characterization of interim efficient allocations*

We introduce one more piece of notation and a simple lemma that provides a formula for expected interim welfare when taxes satisfy incentive compatibility. Given a set of interim welfare weights, $\lambda = \left\{\lambda^i : T^i \to R_+\right\}_{i=1}^N$, let $\lambda^{0i} \equiv \int_{\underline{t}^i}^{\bar{t}^i} \lambda^i(t^i)\, dF^i(t^i)$, denote i's ex ante welfare weight relative to *other players*. Then we define $\Lambda^i(t^i)$ below as a normalized measure of the fraction of i's welfare weight that is concentrated on i's lower types (lower than t^i). [7]

Definition 5. If $\lambda^{0i} > 0$, let $\Lambda^i(t^i) = \frac{1}{\lambda^{0i}} \int_{\underline{t}^i}^{t^i} \lambda^i(s)\, dF^i(s)$. If $\lambda^{0i} = 0$, then $\Lambda^i(t^i) = 0$.

Lemma 4.

$$\begin{aligned}
&\int_{\underline{t}^i}^{\bar{t}^i} \lambda^i(t^i) \left[t^i Q^i(t^i) - A^{0i} - \int_{\underline{t}^i}^{t^i} s\, dQ^i(s) \right] dF^i(t^i) \\
&\quad = \lambda^{0i} \left[\underline{t}^i Q^i(\underline{t}^i) - A^{0i} + \int_{\underline{t}^i}^{\bar{t}^i} \left(\frac{1-\Lambda^i(t^i)}{f^i(t^i)} \right) Q^i(t^i)\, dF^i(t^i) \right].
\end{aligned}$$

Proof. Integrate by parts. □

We can use Lemmas 1, 2, and 4 to provide a more convenient statement of the optimization problem in Theorem 1 where we characterize interim efficiency.

Theorem 2. *There exists $a : T \to R^N$ such that $\eta = (x, a)$ is interim efficient iff there exist non-negative type-dependent welfare weights, $\{\lambda^i\}_{i=1}^N$, where $\sum_i \lambda^{0i} > 0$, and N constants, $\{A^{0i}\}_{i=1}^N$, such that $(x, \{A^{0i}\}_{i=1}^N)$ solves*

$$\max_{\{x:T\to X\}} \sum_{i=1}^N \lambda^{0i} \left[\int_{\underline{t}^i}^{\bar{t}^i} \left(\frac{1-\Lambda^i(t^i)}{f^i(t^i)} \right) Q^i(t^i)\, dF^i(t^i) + \underline{t}^i Q^i(\underline{t}^i) - A^{0i} \right]$$

subject to

$$0 \leqslant \underline{t}^i Q^i(\underline{t}^i) - A^{0i} + \min_{t^i} \left[\int_{\underline{t}^i}^{t^i} Q^i(s)\, ds - U^{0i}(t^i) \right] \textit{for all } i$$

[7] Wilson [41] refers to $\Lambda^i(\cdot)$ as the conditional welfare weights of agent i.

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$$0 \leqslant \sum_{i=1}^{N} \left[\int_{\underline{t}^i}^{\bar{t}^i} \left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)} \right) Q^i(t^i)\, dF^i(t^i) - \underline{t}^i Q^i(\underline{t}^i) + A^{0i} \right] - \int_T C(x(t))\, dF(t)$$

$Q^i(t^i)$ *is non-decreasing in* t^i *for all* i, t^i.

Proof. Follows from Lemmas 1, 2, and 4. The first inequality is individual rationality combined with (IC1). The second inequality is feasibility combined with (IC1). The third condition is (IC2). □

Without individual rationality, this problem simplifies. First, given the linearity of the problem, in the absence of any participation constraints, the (ex ante) welfare weights must all be equal. That is, without loss of generality, $\lambda^{0i} = 1$ for all i. Otherwise, the problem has no solution since one can always improve welfare by arbitrarily large transfers between agents with different ex ante weights. Second, the constant transfers, $\{A^{0i}\}_{i=1}^N$, have no welfare consequences beyond their sum. The following corollary summarizes this.

Corollary 1. *There exists* $a : T \to R^N$ *such that* $\eta = (x, a)$ *is interim efficient* (*without individual rationality*) *iff there exist non-negative type-dependent welfare weights,* $\{\lambda^i\}_{i=1}^N$, *such that for all* i, j, $\lambda^{0i} = \lambda^{0j} = 1$ *and* $x : T \to X$ *solves*:

$$\max_{\{x:T\to X\}} \sum_{i=1}^{N} \left[\int_{\underline{t}^i}^{\bar{t}^i} \left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)} + \frac{1-\Lambda^i(t^i)}{f^i(t^i)} \right) Q^i(t^i)\, dF^i(t^i) - \int_T C(x(t))\, dF(t) \right]$$

subject to

$Q^i(t^i)$ *is non-decreasing in* t^i *for all* i, t^i.

4. The regular case

In this section, we characterize the solution to the problem in Theorem 2, in the case where constraint (IC2) is not binding, and identify conditions under which the solution to this relaxed problem satisfies the missing constraint. When this is true, we refer to the problem as *the regular case*. We adopt a Kuhn–Tucker approach to solving for an optimum.

4.1. Kuhn–Tucker conditions

If we apply the Kuhn–Tucker Theorem to the optimization problem in Theorem 2 the problem can again be restated as follows. In the regular case, there exists a transfer rule $a^* : T \to R^N$ such that (x^*, a^*) is interim efficient if and only if there exists a non-negative system of type-dependent welfare weights, $\{\lambda^i\}_{i=1}^N$, with $\sum_{i=1}^N \lambda^{0i} > 0$, individual multipliers, $\{\rho^i\}_{i=1}^N$, a multiplier, δ, and A^{*0}, such that (x^*, A^{*0}) solves

$$\max_{\{x:T\to X\}, A^0} \sum_{i=1}^{N} \lambda_o^i \left[\int_{\underline{t}^i}^{\bar{t}^i} \left(\frac{1-\Lambda^i(t^i)}{f^i(t^i)} \right) Q^i(t^i)\, dF^i(t^i) + \underline{t}^i Q^i(\underline{t}^i) - A^{0i} \right]$$
$$+ \delta S(x) + \sum_{i=1}^{N} \rho^i \left[\underline{t}^i Q^i(\underline{t}^i) - A^{0i} + \min_{t^i} \left[\int_{\underline{t}^i}^{t^i} Q^i(s)\, ds - U^{0i}(t^i) \right] \right] \tag{2}$$

and

$$
\begin{aligned}
&\rho^i \geqslant 0 \quad \text{for all } i,\\
&0 \leqslant U^i(\underline{t}^i) + \min_{t^i}\left[\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)\right] \quad \text{for all } i,\\
&0 = \rho^i\left[U^i(\underline{t}^i) + \min_{t^i}\left[\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)\right]\right] \quad \text{for all } i,\\
&\delta \geqslant 0,\\
&0 \leqslant S(x),\\
&0 = \delta S(x).
\end{aligned}
$$

The multipliers, $\{\rho^i\}$, δ are for the participation constraints and the resource feasibility constraint, respectively.

4.2. *Solving for ρ and A^{*0}*

Suppose $(\rho, \delta, x^*, A^{*0})$ solves (2) for some λ. First, observe that, at (x^*, A^{*0}), the first order conditions of (2) with respect to A^{0i} are necessary for an optimum, and this implies

$$-\lambda^{0i} - \rho^i + \delta = 0 \quad \text{for all } i. \tag{3}$$

Define $\overline{\lambda} \equiv \max_i\{\lambda^{0i}\}$. Then $\rho^i \geqslant 0$ implies $\delta \geqslant \overline{\lambda} \geqslant \lambda^{0i}$ for all i. Since $\sum_{i=1}^N \lambda^{0i} > 0$, this immediately implies $\delta > 0$ and

$$S(x^*(\cdot)) = 0. \tag{4}$$

From Lemma 3, if $x : T \to X$ satisfies

$$0 \leqslant \sum_{i=1}^N \left[\int_{\underline{t}^i}^{\overline{t}^i} \left(t^i - \frac{1 - F^i(t^i)}{f^i(t^i)}\right) Q^i(t^i)\,dF^i(t^i) - U^i(\underline{t}^i)\right] - \int_T C(x(t))\,dF(t),$$

we can solve for A^{0i} (and hence $a(\cdot)$ as well). Finally,

$$\lambda^{0i} < \overline{\lambda} \Longrightarrow U^i(\underline{t}^i) + \min_{t^i}\left[\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)\right] = 0 \quad \text{for all } i. \tag{5}$$

Eq. (3) implies that we do not need to keep track of each individual rationality constraint separately, and can eliminate the individual rationality multipliers. The other two conditions have implications about both the total amount of taxes and the distribution of taxes.

Eq. (4), implies that the total tax exactly balances the total cost of $x^*(t)$. That is, $\sum_{i=1}^N a^i(t) = C(x^*(t))$ for all t. Hence there is no inefficiency in production (the budget always balances).

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Eq. (5) has implications for the distribution of transfers. First note that if $\lambda^{0i} = \overline{\lambda}$, then A^{*0i} is simply the residual profit from the other agents for whom $A^{*0i} = \underline{t}^i Q^i(\underline{t}^i) + \min_{t^i}[\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)]$. Second, if

$$\sum_{i=1}^{N}\left[\int_{\underline{t}^i}^{\overline{t}^i}\left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)}\right)Q^i(t^i)\,dF^i(t^i) - \min_{t^i}\left[\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)\right]\right] = \int_T C(x(t))\,dF(t),$$

then $A^{*0i} = \underline{t}^i Q^i(\underline{t}^i) + \min_{t^i}[\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)]$ for all i, and this must hold if $\delta > \overline{\lambda}$.

Having dispensed with A_0^i and ρ^i, it is a straightforward exercise to substitute and rearrange terms to give the following theorem that completely characterizes interim efficiency in the regular case. In order to write a clean expression that covers the many applications where the worst type [8] is not $\underline{t}^i$, we use the indicator function $I^i(t^i)$, where

$$\begin{aligned} I^i(t^i) &= 1 \quad \text{if } t^i < \arg\min_{t^i}\left[t^i Q^i(t^i) - A^{0i} - \int_{\underline{t}^i}^{t^i} s\,dQ^i(s) - U^{0i}(t^i)\right] \\ &= 0 \quad \text{if } t^i \geqslant \arg\min_{t^i}\left[t^i Q^i(t^i) - A^{0i} - \int_{\underline{t}^i}^{t^i} s\,dQ^i(s) - U^{0i}(t^i)\right]. \end{aligned}$$

Theorem 3. *For the regular case $\exists\, a^* : T \to R^N$ such that (x^*, a^*) is interim efficient if and only if there exist non-negative $\{\lambda^i\}_{i=1}^N$ with $\sum_{i=1}^N \lambda^{0i} > 0$ and $\delta \geqslant \overline{\lambda}$ such that*

$$x^* \in \arg\max_{x:T\to X}\left\{\begin{array}{c}\sum_i \int_{\underline{t}^i}^{\overline{t}^i}\left[t^i - \frac{1-F^i(t^i)}{f^i(t^i)} + \frac{\lambda^{0i}}{\delta}\frac{1-\Lambda^i(t^i)}{f^i(t^i)} + \left(1-\frac{\lambda^{0i}}{\delta}\right)\frac{I^i(t^i)}{f^i(t^i)}\right]Q^i(t^i)\,dF^i(t^i) \\ -\int_T C(x(t))\,dF(t)\end{array}\right\}, \tag{6}$$

$$\begin{aligned} 0 \leqslant \sum_{i=1}^{N}&\left[\int_{\underline{t}^i}^{\overline{t}^i}\left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)}\right)Q^i(t^i)\,dF^i(t^i)\right. \\ &\left.+\min_{t^i}\left(\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)\right)\right] - \int_T C(x(t))\,dF, \end{aligned} \tag{7}$$

$$0 = (\delta - \overline{\lambda})\left\{\begin{array}{c}\sum_i\left[\int_{\underline{t}^i}^{\overline{t}^i}\left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)}\right)Q^i(t^i)\,dF^i(t^i) + \min_{t^i}\left(\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)\right)\right] \\ -\int_T C(x(t))\,dF\end{array}\right\}. \tag{8}$$

Note that this requires $x^*(\cdot)$ and δ to simultaneously solve (6)–(8) rather than maximizing (6) subject to (7) and (8).

[8] The worst type is defined by $argmin_{t^i}\,[t^i Q^i(t^i) - A^{0i} - \int_{\underline{t}^i}^{t^i} s\,dQ^i(s) - U^{0i}(t^i)]$.

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4.3. Virtual valuations

The term of the maximand of (6) in large square brackets can be written as

$$W^i(t^i, \lambda^i, \delta) \equiv t^i - \left(1 - \frac{\lambda^{0i}}{\delta}\right) \frac{1 - F^i(t^i) - I^i(t^i)}{f^i(t^i)} - \frac{\lambda^{0i}}{\delta} \frac{\Lambda^i(t^i) - F^i(t^i)}{f^i(t^i)}.$$

We call this the *virtual valuation* of type t^i after Myerson [32] and others (Wilson [41], Gresik [12], and Ledyard and Palfrey [24,25]). The virtual valuation is equal to the player's type, t^i, with adjustments due to two factors. The first adjustment, $-(1 - \frac{\lambda^{0i}}{\delta})\frac{1-F^i(t^i)-I^i(t^i)}{f^i(t^i)}$, is for information rents resulting from binding participation constraints. This corresponds to the relevant terms in Myerson and Satterthwaite [33] where λ^i equals the constant function **1** for both buyer and seller. In their notation the adjustment to the buyer's virtual valuation equals $-\alpha\frac{1-F^i(t^i)}{f^i(t^i)}$ and the adjustment to the seller's virtual valuation equals $\alpha\frac{F^i(t^i)}{f^i(t^i)}$.

The second adjustment is due to possible *desirable* distortions arising from redistribution of income, which occurs if welfare weights are type dependent. This adjustment is given by the expression $-\frac{\lambda^{0i}}{\delta}\frac{\Lambda^i(t^i)-F^i(t^i)}{f^i(t^i)}$.

If participation constraints are not binding anywhere, then $\delta = \lambda^{0i}$ for all *i*. In this case the first adjustment term disappears entirely and the whole expression for virtual valuations reduces to $W^i(t^i, \lambda^i, \delta) \equiv t^i - \frac{\Lambda^i(t^i)-F^i(t^i)}{f^i(t^i)}$, so differences between actual and virtual valuations are driven entirely by the type dependent welfare weights, reflecting distributive considerations. If there are no distributive considerations, i.e., $\lambda^i(t^i) = 1$ for all *i* and t^i, then $\Lambda^i(t^i) = F^i(t^i)$ and $W^i(t^i, \lambda^i, \delta) \equiv t^i$.

4.4. Sufficient conditions for regularity

The solution to the regular case was obtained by simply dropping the constraint that Q^i be non-decreasing, so the question is: When is the solution to this "relaxed" problem also a solution to the original problem? [9] A complete answer to this question would give a full characterization of the regular case. A partial answer is easier to find. Specifically, a sufficient condition for Q^i to be non-decreasing $\forall i, t^i$ is that $\frac{\partial W^i}{\partial t^i} \geqslant 0$, for all t^i, *i* and for all $\delta \geqslant \overline{\lambda}$. That is, virtual valuations are monotone in type. As Gresik [12] and Ledyard and Palfrey [24,25] recognized, this boils down to a joint condition on priors F_i *and* welfare weights λ. The standard condition (i.e. without welfare weights or participation constraints), that $t^i - \frac{1-F^i(t^i)}{f^i(t^i)}$ be increasing in t^i for all *i*, is neither necessary nor sufficient. For example if F^i is uniform on [0, 1] then

$$W^i(t^i) = t^i - \frac{1 - F^i(t^i)}{f^i(t^i)} + \frac{\lambda^{0i}}{\delta}\left(\frac{1 - \Lambda^i(t^i)}{f^i(t^i)}\right)$$

[9] In this section we only consider buyers, for whom $I^i(t^i) = 0$ for all t^i. The cases where $I^i(t^i)$ is not necessarily 0 (e.g., sellers) are similar.

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$$
\begin{aligned}
&= t^i - [1 - F^i(t^i)] + \frac{\lambda^{0i}}{\delta}[1 - \Lambda^i(t^i)] \\
&= 2t^i - 1 - \frac{\lambda^{0i}}{\delta}\left[1 - \int_{\underline{t}^i}^{t^i} \lambda^i(s)\, dF^i(s)\right].
\end{aligned}
$$

So $\frac{\partial W^i}{\partial t^i} = 2 - \frac{\lambda^{0i}}{\delta}\lambda^i(t^i)$. For the special case of constant welfare weights, say $\lambda = 1$, this implies $\frac{\partial W^i}{\partial t^i} = 2 - \frac{1}{\delta} > 0$ since $\delta > 1$, so the solution to the relaxed problem for the uniform case is always optimal.[10] But for interim efficiency, which allows for non-constant $\lambda(t^i)$, one may need further restrictions in order to satisfy the second order conditions of the full optimization problem. For example, in the uniform case described above, the solution to the relaxed problem satisfies the second order conditions of the full problem if and only if $\lambda^i(t^i) \leqslant 2\delta$ for all i, t^i.

If the standard condition holds ($t^i - \frac{1-F^i(t^i)}{f^i(t^i)}$ increasing in t^i for all i), then a sufficient condition for the regular case is that $\frac{\partial W^i}{\partial t^i} \geqslant 0$, for all t^i, i when $\delta = \overline{\lambda}$. This is satisfied if $t^i + \frac{F^i(t^i) - \Lambda^i(t^i)}{f^i(t^i)}$ is increasing in t^i for all i.

When $\frac{\partial W^i}{\partial t^i}(\widehat{t^i}) < 0$ for some $i, \widehat{t^i}$, we are in the irregular case. Here, the constrained optimal solution can be obtained by a procedure known as "ironing" [13,38]; that is, Q^i must be constant over some interval, which results in flat regions, sometimes referred to as bunching of types. This raises a question of which interim efficient mechanisms are missed by the algorithm based on virtual valuations.

5. Applications

We next turn to applications of the characterization of interim efficient mechanisms in several different regular economic environments. Summarizing the previous section, a specific application consists of a specification of

$$N, X, C(x), \{T^i, F^i, q^i : X \to \Re, U^{0i} : T^i \to \Re\}_{i=1}^N.$$

To find a specific interim efficient allocation for such an environment, one specifies a collection of type-contingent welfare weights, $\{\lambda^i : T^i \to \Re^+\}_{i=1}^N$ and applies the techniques outlined in the previous section.

Following Theorem 3, interim efficient mechanisms in these settings can be derived by simply modifying the original first best problem by replacing the valuation t^i, with the virtual valuation $W^i(t^i, \lambda^i, \delta)$, for suitably chosen δ. Thus it is much like a classic welfare optimization problem, where expected *virtual* welfare (minus costs) is the criterion function, with the added complication of IR constraints. This leads to a natural algorithm, using virtual valuations in the place of the actual private valuations.

Step 1: Set $\delta = \overline{\lambda}$, and for each t let $x^*_\delta(t)$ solve $\max_{x \in X} \sum_{i=1}^N W^i(t^i, \lambda^i, \delta) q^i(x) - C(x)$. If

$$
\begin{aligned}
&\sum_i \int_{\underline{t}^i}^{\overline{t}^i} \left(t^i - \frac{1 - F^i(t^i)}{f^i(t^i)}\right) q^i\left(x^*_\delta(t)\right) dF(t \mid t^i) + \min_{t^i}\left[\int_{\underline{t}^i}^{t^i} Q^i(s)\, ds - U^{0i}(t^i)\right] \\
&\quad - \int_T C\left(x^*_\delta(t)\right) dF(t) \geqslant 0,
\end{aligned}
$$

this is the solution, and go to step 4. If not, then

[10] The case of constant welfare weights corresponds ex ante efficiency.

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Step 2: For every $\delta > \overline{\lambda}$, for each t let $x^*_\delta(t)$ solve $\max_{x \in X} \sum_{i=1}^N W^i(t^i, \lambda^i, \delta)q^i(x) - C(x)$.
Step 3: Find the minimum value of δ such that

$$\sum_i \int_{\underline{t}^i}^{\overline{t}^i} \left(t^i - \frac{1 - F^i(t^i)}{f^i(t^i)}\right) q^i\left(x^*_\delta(t)\right) dF(t \mid t^i) + \min_{t^i} \left[\int_{\underline{t}^i}^{t^i} Q^i(s)\, ds - U^{0i}(t^i)\right] - \int_T C\left(x^*_\delta(t)\right) dF(t) \geqslant 0.$$

Step 4: Calculate $a^*(t)$ using the formula in the proof of Lemma 2. The solution is $x^*_\delta(t)$.

We consider two applications in detail in this section: public goods and auctions. Other applications can be found in Ledyard and Palfrey [27]. Part of the point of this section is to illustrate how all of these models are contained as special cases of the general framework in this paper. We also develop some new results about interim efficient mechanisms for excludable public goods and auctions.

5.1. *Public goods*

Properties of interim efficient public good mechanisms differ depending on whether exclusion is feasible and whether IR constraints are included in the formulation of the problem. We describe these dependencies below. In this subsection there are no sellers and we assume the regular case. Hence virtual valuations reduce to

$$W^i(t^i, \lambda^i, \delta) \equiv t^i - \frac{1 - F^i(t^i)}{f^i(t^i)} + \frac{\lambda^{0i}}{\delta} \frac{1 - \Lambda^i(t^i)}{f^i(t^i)}$$

and W^i is non-decreasing in t^i.

5.1.1. *Pure public goods*

Without IR constraints: Without IR constraints, and with type-independent welfare weights, the efficient outcome is to produce the first best output and tax each agent the balancing incentive taxes described in proof of Lemma 2 [1]. Incentive constraints are not binding. However, with non-constant welfare weights the story is much different. In that case, redistribution of the private good across types has social value and incentive constraints are binding, resulting in distortions from the first best output.

In the notation of this paper, the pure public goods model is $X = [0, 1]$, $C(x) = Kx$, $q^i(x) = x$. Since individual rationality is not required $\delta = \overline{\lambda}$. Hence, in the regular case, given welfare weights, $\lambda^i : T^i \longrightarrow \Re^+$, the interim efficient mechanism is characterized by: $x^*(t) \in \arg\max_x \left(\sum_i t^i + \frac{F^i(t^i) - \Lambda^i(t^i)}{f^i(t^i)} - K\right) x$. That is, the efficient public decision *always* involves a simple cost benefit calculation: produce $x = 1$ if and only if the sum of the virtual valuations exceeds the cost of production; otherwise, produce $x = 0$.

The nature of distortion from first best depends in systematic ways on the type dependent welfare weights. Suppose for example that $q^i(t^i) = q(t^i)$ for all i, q is concave, increasing, and C is convex and increasing. Then a first-best decision, x^0, satisfies $\sum_i t^i \frac{\partial q(x^0)}{\partial x} = \frac{\partial C(x^0)}{\partial x}$. For interim efficient mechanisms, given a set of welfare weights, a necessary condition for interim

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efficiency in the regular case is

$$\sum_{i=1}^{N} W^i(t^i, \lambda^i)\,\frac{\partial q(x^*)}{\partial x} = \frac{\partial C(x^*)}{\partial x}.$$

Therefore, if $W^i(t^i, \lambda^i) > t^i$ for all t then $x^*_\lambda(t) \geqslant x^0(t)$ and there is more production than the ex ante efficient mechanism. [11] Indeed, $W^i(t^i, \lambda^i) > t^i$ occurs, for example, if $\lambda^i(t^i)$ is increasing in t^i. That is, when higher types are more heavily weighted than lower types, over-production is a more efficient way to relax incentive compatibility constraints than transfers, a^i. The economic intuition behind this result is the following. First, since higher types are weighted more heavily, welfare is increased either by shifting taxes from high types to low types or by producing the public good more often. However, the only way to shift the tax burden from higher types to lower types, without violating incentive compatibility or feasibility, is to produce the public good less often, which would make high types worse off. This intuition does not depend on the linearity of q^i in x or the linearity of the production technology.

With IR constraints: With individual rationality constraints, two results follow quickly for regular environments. For simplicity, we deal only with the case of constant welfare weights (ex ante efficiency), but the same results hold with general welfare weights.

The first result is for the ex ante case where λ is constant in type. With binding participation constraints,

$$W^i(t^i, \lambda^i, \delta) \equiv t^i - \frac{1 - F^i(t^i)}{f^i(t^i)} + \frac{\lambda^{0i}}{\delta}\,\frac{1 - \Lambda^i(t^i)}{f^i(t^i)},$$

no longer reduces to $W^i(t^i, \lambda^i, \delta) \equiv t^i$, because $\delta > \overline{\lambda}$. Instead, one gets

$$W^i(t^i, \lambda^i, \delta) \equiv t^i - \left(1 - \frac{\lambda^{0i}}{\delta}\right)\frac{1 - F^i(t^i)}{f^i(t^i)}.$$

Therefore virtual valuations are lower, so the efficient choice of x is *always lower* with individual rationality constraints than without. [12] For example, suppose $X = [0, 1]$, $C(x) = Kx$ and $q^i(x) = x$. Then for some realizations of t such that $\sum_i t^i - K$ is positive, but not very large, it will be necessary to produce zero because there is not enough surplus to cover incentive costs without violating individual rationality.

The second observation is that individual rationality implies that interim efficient public good production collapses with large N. That is, per capita output must go to 0, except in the uninteresting case where positive production is optimal for all realizations of t. The intuition is simple, and a detailed argument appears in Ledyard and Palfrey [25]. [13] In fact, *all* interim efficient mechanisms have this property. The reason is that $Q^i(t^i)$ must converge to a constant, say $\overline{q}$, that is independent of i and t^i, because with many agents one single agent's report can have only an infinitesimal effect on per capita production (assuming t^i is bounded above). Incentive compatibility then implies that $A^i(t^i)$ must also converge to a constant, which, by resource feasibility equals the average

[11] The distortion leads to less production if $W^i(t^i, \lambda^i) < t^i$ for all t.

[12] This is not true generally for interim efficient mechanisms where weights can be type dependent. It will be true if $\frac{\partial W^i}{\partial \delta} = -\frac{\lambda^{0i}}{\delta^2}\frac{1-\Lambda^i(t^i)}{f^i(t^i)} \leqslant 0$ but this is guaranteed if and only if $\Lambda^i(t^i) \leqslant 1$ for all t^i.

[13] Güth and Hellwig [14] and Mailath and Postlewaite [29] establish similar results.

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cost share, call it c. Thus the individual rationality constraint becomes $t^i\overline{q} \geqslant c$ for all i and t^i. But if $\underline{t}^i < 0$ for some i, and $c > 0$, this implies $\overline{q} = 0$. So, unless it is individually rational to produce a positive quantity of the public good, with equal taxation, for all realizations of t, public good production goes to zero as $N \to \infty$.

5.1.2. Excludable public goods

Without IR constraints: An excludable public good is one for which i's consumption of the good is allowed to be any y^i such that $0 \leqslant y^i \leqslant x$. So $U^i = t^i q^i(y^i) - a^i, x \in R_+$. Here, $(x, y^i, \ldots, y^N)$ is feasible if and only if $0 \leqslant y^i \leqslant x$ for $i = 1, \ldots, N$.

The social decision for an interim efficient mechanism thus solves

$$\max_{(x,y^i,\ldots,y^N)} \sum_{i=1}^{N} W^i(t^i, \lambda^i) q^i(y^i) - C(x)$$

subject to $x \in R_+$, $0 \leqslant y^i \leqslant x$.

So assuming $\frac{dq^i}{dy^i} \geqslant 0$, and second order conditions are satisfied, interim efficient allocations satisfy, for each t,

$$x^*(t) \in \arg\max_x \sum_{i=1}^{N} \max\left\{W^i(t^i, \lambda^i), 0\right\} q^i(x) - C(x)$$

and $y^i = x$ iff $W^i(t^i, \lambda^i) \geqslant 0$.

For ex ante efficiency, $W^i(t^i) = t^i$. So

$$x^* \in \arg\max_x \sum_{i=1}^{N} \max\left\{t^i, 0\right\} q^i(x) - C(x)$$

and $y^i = x$ iff $t^i \geqslant 0$. Note that if $\underline{t}^i \geqslant 0$, then $y^i = x$ always and there is no difference between the ex ante efficient mechanisms in the pure public good case and the excludable case. The threat of exclusion provides no help in relaxing incentive constraints, simply because incentive constraints are not binding to begin with. However, there can be a difference for interim efficiency if welfare weights are type dependent.

For *interim efficiency*, $W^i = t^i - \frac{\Lambda^i(t^i) - F^i(t^i)}{f^i(t^i)}$. So $y^i = x$ iff $t^i \geqslant \frac{\Lambda^i(t^i) - F^i(t^i)}{f^i(t^i)}$ and $x^*(t) \in \arg\max_x \sum_i \max\left\{t^i - \frac{\Lambda^i(t^i) - F^i(t^i)}{f^i(t^i)}, 0\right\} q^i(x) - C(x)$. It follows that if the welfare weights favor low types then $\Lambda^i(t^i) - F^i(t^i) > 0$ and there is lower production of x and more types are excluded than under the ex ante efficient mechanism. If the weights favor high types then $\Lambda^i(t^i) - F^i(t^i) < 0$ and there is higher production and less exclusion relative to the ex ante efficient mechanism. [14]

At first blush, this may seem surprising. Excluding less often than the ex ante efficient mechanism means that some agents with negative valuations are forced to consume the public good. Excluding them involves no resource costs, and makes them better off, so how can this possibly be efficient? The answer is that when weights favor high types, then it is optimal to tax low types

[14] Coughlan [5] studies excludable public goods with congestion costs and no IR constraint. The results are similar, with an additional adjustment term for the congestion externality.

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as much as possible. But if *all* negative types are always excluded, then they must all pay the same tax, by incentive compatibility, but (interim) efficiency may in some cases be improved by discriminatory taxation on negative types. Thus, the only way to have variation in the taxes of negative types is to have forced inclusion.

With IR constraints: First consider the simple ex ante case, with equal weights.[15] Applying our techniques, in regular environments, the ex ante optimal mechanism solves the following problem, for suitably chosen $\delta > 1$:

$$\max_{x \geqslant 0} \sum_{i=1}^{N} \max\left\{t^i - \frac{\delta-1}{\delta}\frac{1-F(t^i)}{f^i(t^i)}, 0\right\} q^i(x) - C(x)$$

$$\text{and } y^i = x \text{ iff } t^i \geqslant \frac{\delta-1}{\delta}\frac{1-F^i(t^i)}{f^i(t^i)}.$$

For any value of δ, denote the solution to the above problem by $(x_\delta, y_\delta^1, \ldots y_\delta^N)$. The multiplier δ is the minimum value greater than or equal to 1, such that

$$\sum_{i=1}^{N} \int_{T^i} \left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)}\right) q^i(y_\delta^i)\, dF^i(t^i) \geqslant \int_T C(x_\delta(t))\, dF(t).$$

Thus, if individual rationality is binding, $\delta > 1$, so it may be ex ante efficient to exclude some types even though ex post it would be efficient to include them. This occurs if there exist some $t^i > 0 > t^i - \frac{\delta-1}{\delta}\frac{1-F^i(t^i)}{f^i(t^i)}$. So exclusion can provide help in relaxing the individual rationality constraints. However, note that this exclusion does not always happen even if participation constraints bind. If $\underline{t}^i f(\underline{t}^i) \geqslant 1$ and we are in the regular case then $W^i(t^i, \delta) \geqslant 0$ for all t^i and i is never excluded in an ex ante efficient mechanism.[16]

The limiting case with many agents is of interest, to answer the question of whether exclusion provides a way around the negative result for pure public goods. The answer is generally yes, and we illustrate it first with ex ante efficient mechanisms, for the linear symmetric case where $q^i(y^i) = y^i$, $C(x) = Nkx$, $k \in (0, 1)$, $F^i = F^j$ for all i, j, and $x \in [0, 1]$.

When the IR constraint is binding, $\delta > 1$. Let t_δ^0 solve $W^i(t_\delta^0, \delta) = 0$, or equivalently, $t_\delta^0 - \frac{\delta-1}{\delta}\frac{1-F^i(t_\delta^0)}{f^i(t_\delta^0)} = 0$. That is, t_δ^0 is the boundary type separating those who are excluded from those who are not excluded, given δ. By symmetry, the individual rationality constraint can be written for a representative agent, and reduces to $\int_{t_\delta^0}^{\bar{t}} \left(t - \frac{1-F(t)}{f(t)}\right) dF(t) \geqslant k$ when $x \to 1$. So if there is a value of $\delta > 1$ such that $t_\delta^0(1 - F(t_\delta^0)) \geqslant k$, then positive production of the public good occurs even as $N \to \infty$, and some types will be excluded (and, by IR, pay no tax). In the limit, IR is binding on the lowest *included type*, so $t_\delta^0(1 - F(t_\delta^0)) = k$. Thus the efficient solution is characterized by a flat user fee equal to t_δ^0, which just covers the cost of production.

Does interim efficiency change these properties as $N \to \infty$? Without individual rationality constraints, $W^i = t^i - \frac{\Lambda^i(t^i) - F^i(t^i)}{f^i(t^i)}$ and i is excluded iff $W^i(t^i, \lambda^i) < 0$. Let t^{0i} be the solution to $W^i(t^{0i}, \lambda^i) = 0$ and consider the regular case where $\frac{\partial W^i(t^i, \lambda^i)}{\partial t^i} \geqslant 0$. Now $x \to 1$ as $N \to \infty$ iff

[15] See Norman [34] for a detailed analysis. Hellwig [16] focuses on limiting results for many agents. Cornelli [4] and Schmitz [39] examine profit maximization for a monopolist in a more specialized setting.

[16] For the uniform distribution $\underline{t} f(\underline{t}) \geqslant 1$ iff $\underline{t} \geqslant (1/2)\bar{t}$.

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$E[\max\{W^i, 0\}] \geqslant k$. That is $x \to 1$ as $N \to \infty$ iff $t^{0i}(\Lambda^i(t^{0i}) - F^i(t^{0i})) + \int_{t^{0i}}^{\bar{t}} (s\lambda(s)\, dF(t^i) \geqslant k$. So there can be positive production of the public good. Also if low types are favored, (that is, λ is decreasing in type), then relative to ex ante efficiency there will be more exclusion and less production. The opposite is true if high types are favored.

Next consider the limiting solution with participation constraints. For suitable δ, an interim efficient mechanism excludes all $t \leqslant \widehat{t}$ where $\widehat{t}$ satisfies

$$\widehat{t} - \frac{1 - F(\widehat{t})}{f(\widehat{t})} + \frac{1}{\delta}\frac{1 - \Lambda(\widehat{t})}{f(\widehat{t})} = 0.$$

To determine the suitable δ, note that the participation constraint is

$$\sum_{i=1}^{N} \int_{T^i} \left(t^i - \frac{1 - F^i(t^i)}{f^i(t^i)}\right) Q_\delta^i(t^i)\, dF^i(t^i) \geqslant \int_T C(x_\delta(t))\, dF(t).$$

By symmetry, this reduces to

$$\int_{T^i} \left(t^i - \frac{1 - F^i(t^i)}{f^i(t^i)}\right) Q_\delta^i(t^i)\, dF^i(t^i) \geqslant \frac{1}{N}\int_T C(x_\delta(t))\, dF(t).$$

In the limit, either $x(t) = 1$ for all t or $x(t) = 0$ for all t depending on whether $\int_{\underline{t}}^{\bar{t}} W^i(t, \lambda, \delta)\, dF^i(t) \gtrless k$. In the first case, individual rationality is satisfied in the limit if and only if

$$\widehat{t}(1 - F(\widehat{t})) \geqslant k.$$

But note that this condition is independent of the welfare weights, and is precisely the same condition we had for ex ante efficiency. Therefore, the limiting solution is the same for all welfare weights. That is, in the limit, there is a unique interim efficient mechanism, characterized by the exclusion cutoff type t^0 satisfying $t^0(1 - F(t^0)) = k$, if it exists.[17] If there is no solution to t^0, then in the limit there is either no production (if $t(1 - F(t)) < k$ for all t) or there is always production and no exclusion (if $t(1 - F(t)) > k$ for all t); these two cases also correspond to a cutoff, either 0 or ∞. Also note that in both of these boundary solutions the outcome function is "as if" the public good were not excludable. Any of these cutoff solutions can be implemented in dominant strategies, where a "user fee" equal to $\frac{k}{1-F(t^0)}$ is posted and charged to anyone who wishes to enjoy the use of the public good. Hence (IC2) is not binding, so the restriction to regular mechanisms is not needed in the limit.

Proposition 1. *In the symmetric case with excludable public goods, where $q^i(y^i) = y^i$, $C(x) = Nkx$, $F^i = F^j$ for all i, j, and $x \in [0, 1]$, all interim efficient individually rational mechanisms are asymptotically equivalent, and can be implemented with a simple user fee that will exactly cover the cost of production.*[18]

As an example, suppose F is uniform on [0, 1]. Then $W^i(t, \lambda, \delta) = 2t - 1 + \frac{1}{\delta}(1 - \Lambda(t))$ and $(1 - t^0) = k$. If $k > \frac{1}{4}$ then $x = 0$ as $N \to \infty$. If $k \leqslant \frac{1}{4}$, then $x \to 1$ as $N \to \infty$,

[17] If there is more than one solution to $t^0(1 - F(t^0)) = k$, then the minimum solution is optimal.

[18] The limiting value of the multiplier on the participation constraint may differ according to the welfare weights, since it must satisfy $t^0 - \frac{1-F(t^0)}{f(t^0)} + \frac{1}{\delta}\frac{1-\Lambda(t^0)}{f(t^0)} = 0$.

$t^0 = \frac{1}{2} - \frac{1}{2}\sqrt{1-4k}$ and $\frac{1-\delta}{\delta} = \frac{2t^0 - \Lambda(t^0)}{1-\Lambda(t^0)}$. $W^i(t, \lambda, \delta) = 2t - 1 + \left(\frac{\sqrt{1-4k}}{1-\Lambda(t^0)}\right)(1 - \Lambda(t^0))$. And $\frac{dW^i}{dt}(t^0) = 2 - \left(\frac{\sqrt{1-4k}}{1-\Lambda(t^0)}\right)\lambda(t^0)$. Notice that, as is true in general, t^0 does not depend on the welfare weights. When the individual rationality constraints are binding as $N \to \infty$, the cut-off point for exclusion, t^0, is such that if all who are not excluded pay an equal "user fee" equal to t^0 then costs are exactly covered.

5.2. *Private goods*

Myerson [32], Myerson and Satterthwaite [33], Wilson [42], Cramton et al. [6] and others have studied ex ante efficient mechanisms for linear private good environments. [19] In our notation, for all of these settings,

$$X = \left\{ x \in \Re^N \,\middle|\, \sum_{i-1}^{N} x^i \leqslant J \right\},$$
$$U^i = t^i q^i - a^i,$$
$$C(x) = 0,$$

where J is the quantity of private good available.

In the exchange environments considered here, the set of agents is divided into two categories, buyers and sellers. Buyers are assumed to have no endowment of the good to be exchanged, but unlimited amounts of the transferable utility good. The buyers and sellers have $q^i(x) = x^i$. Each seller owns one unit of the good to be exchanged and this is reflected in their participation constraints, as described in earlier sections. These problems neatly divide themselves into specific applications, depending on the number of buyers and sellers. We distinguish the following four applications in this way:

1. *Bargaining*: 1 buyer and 1 seller.
2. *Markets*: $I > 1$ buyers and $J > 1$ sellers.
3. *Auctions*: I buyers and 1 seller (or 1 buyer and J sellers).
4. *Assignment*: I buyers and 0 sellers.

Ledyard and Palfrey [27] treats all of these cases. Because of space constraints, we focus on auctions here.

5.2.1. *Auctions: many buyers and one seller (or one buyer and many sellers)*

The problem of designing revenue-maximizing auctions when buyers have independent private values was initiated by Vickrey [40], but not solved until 1981, when three papers were published almost simultaneously by Harris and Raviv [15], Myerson [32], and Riley and Samuelson [36].

Here, we address a more general version of the problem, characterizing all *interim efficient* auctions. The expected revenue maximizing auction [20] arises as a special case, which corresponds in our framework to setting all the buyers' welfare weights to 0, and setting the seller's welfare weights to a positive constant. For that special case, it is already well known that the optimal

[19] Gresik [12] and Wilson [41] consider interim efficient mechanisms in private good settings.

[20] Formally, this is only revenue maximization if the seller's type is 0. It would be more precise to call this expected profit maximization, where the seller's type can be viewed as the cost.

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mechanism can be implemented many simple ways, such as a second price auction with a publicly announced reserve bid, where the reserve bid is a function of the seller's type.

In the general case with type-dependent seller weights, the implementation of optimal mechanisms by auctions can be much more complicated, in particular, secret reserve bids and bid-dependent reserve bids may be optimal. This is true, even if the buyer welfare weights are equal to 0. If buyer welfare weights are positive, the problem is even further complicated. At the opposite extreme, where all the weight is on the buyers' welfare, the problem becomes equivalent to the general assignment problem, which is analyzed in the next section.

Denote the seller by s, and the buyers by $i = 1, \dots, n$. Recall that, for the buyers

$$W^i(t^i, \lambda^i, \delta) \equiv t^i - \frac{1 - F^i(t^i)}{f^i(t^i)} + \frac{\lambda^{0i}}{\delta} \frac{1 - \Lambda^i(t^i)}{f^i(t^i)}$$

and for the sellers

$$W^S(t^S, \lambda^S, \delta) \equiv t^S + \frac{F^S(t^S)}{f^S(t^S)} - \frac{\lambda^{0i}}{\delta} \frac{\Lambda^S(t^S)}{f^S(t^S)}.$$

Therefore, it follows from Theorem 3 that $\exists a^*$ such that (x^*, a^*) is an interim efficient auction if and only if there exist non-negative functions $\lambda^s(t^s)$, $\{\lambda^i(t^i)\}_{i \in I}$, not all 0, and $\delta \geqslant \overline{\lambda}$ such that x^* maximizes

$$\int_{\underline{t}^s}^{\overline{t}^s} \left(t^s + \frac{F^s(t^s)}{f^s(t^s)} - \frac{\lambda^{0s}}{\delta} \frac{\Lambda^s(t^s)}{f^i(t^s)} \right) Q^s(t^s)\, dF^s(t^s)$$
$$+ \sum_{i=1}^{n} \left[\int_{\underline{t}^i}^{\overline{t}^i} \left(t^i - \frac{1 - F^i(t^i)}{f^i(t^i)} + \frac{\lambda^{0i}}{\delta} \frac{1 - \Lambda^s(t^s)}{f^i(t^s)} \right) Q^i(t^i)\, dF^i(t^i) \right]$$

subject to $\sum_i x^i \leqslant 1$

and

$$0 \leqslant \int_{\underline{t}^i}^{\overline{t}^i} \left(t^s + \frac{F^s(t^s)}{f^s(t^s)} \right) Q^s(t^s)\, dF^s(t^s)$$
$$+ \sum_{i=1}^{n} \int_{\underline{t}^i}^{\overline{t}^i} \left(t^i - \frac{1 - F^i(t^i)}{f^i(t^i)} \right) Q^i(t^i)\, dF^i(t^i)$$

$$0 \leqslant \delta - \overline{\lambda},$$

$$0 = (\delta - \overline{\lambda}) \left\{ \int_{\underline{t}^s}^{\overline{t}^s} \left(t^s + \frac{F^s(t^s)}{f^s(t^s)} \right) Q^s(t^s)\, dF^s(t^s) \right.$$
$$\left. + \sum_{i=1}^{n} \int_{\underline{t}^i}^{\overline{t}^i} \left(t^i - \frac{1 - F^i(t^i)}{f^i(t^i)} \right) Q^i(t^i)\, dF^i(t^i) \right\}.$$

We next show how the familiar revenue maximization problem falls out of our framework.

Revenue maximization: For revenue maximization, assume that $\lambda^i(t^i) = 0$ for all i and t^i, and $\lambda^s(t^s) = 1$ for all t^s. This implies that welfare is maximized by maximizing the expected surplus to the seller. Assuming participation constraints are not binding on any seller type, the inequality constraint is slack, so $\delta = 1$. From the characterization earlier, $\exists a^*$ such that (x^*, a^*) is an interim

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efficient auction if and only if x^* maximizes

$$\int_{\underline{t}^s}^{\bar{t}^s} t^s Q^s(t^s)\, dF^s(t^s) + \sum_{i=1}^{n} \left[\int_{\underline{t}^i}^{\bar{t}^i} \left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)} \right) Q^i(t^i)\, dF^i(t^i) \right]$$

$$\text{subject to } \sum_i x^i \leqslant 1.$$

Assuming we are in the regular case, this gives us the following well-known solution.

Proposition 2. *Pick any buyer* $i^* \in \arg\max_i \left\{ t^i - \frac{1-F^i(t^i)}{f^i(t^i)} \right\}$. *If* $t^{i^*} - \frac{1-F(t^{i^*})}{f(t^{i^*})} \geqslant t^s$, *then sell to* i^* at price $t^{i^*} - \frac{1-F(t^{i^*})}{f(t^{i^*})}$. *Otherwise do not sell. This can be implemented by announcing a reserve bid for each bidder,* $\tilde{t}^i$, *defined by* $\tilde{t}^i - \frac{1-F(\tilde{t}^i)}{f(\tilde{t}^i)} = t^s$ *and then holding a first price sealed bid auction.*

Interim efficient auctions that are not revenue maximizing: We next consider the case where the welfare weights are still concentrated on the seller, but the welfare weights are not the same for all seller types, so that $\lambda^i = 0$ for $i = 1, \dots, n$ as before, but $\lambda^s(t^s)$ is not constant.[21] This case is more interesting for two reasons. First, $F^s - \Lambda^s \neq 0$, so there will be cross subsidization of seller types. Second it is possible that $\delta > 1$, if there is sufficient cross subsidization that individual rationality is binding on some seller types. This could arise, for example, if some sellers whose valuations are in the support of the buyers' valuations are earning 0 profits.

Without loss of generality, we can normalize $\lambda^s(t^s)$ so that $\lambda^{0s} = 1$. By doing so, for suitably chosen δ the maximand reduces to:

$$\int_{\underline{t}^s}^{\bar{t}^s} \left(t^s + \frac{F^s(t^s)}{f^s(t^s)} - \frac{1}{\delta} \frac{\Lambda^s(t^s)}{f^i(t^s)} \right) Q^s(t^s)\, dF^s(t^s) + \sum_{i=1}^{n} \left[\int_{\underline{t}^i}^{\bar{t}^i} \left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)} \right) Q^i(t^i)\, dF^i(t^i) \right]$$

or

$$\int_T \left[\left(t^s + \frac{F^s(t^s)}{f^s(t^s)} - \frac{1}{\delta} \frac{\Lambda^s(t^s)}{f^i(t^s)} \right) q^s(t) + \sum_{i=1}^{n} \left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)} \right) q^i(t) \right] dF^s(t^s).$$

Provided $t^s + \frac{F^s(t^s)}{f^s(t^s)} - \frac{1}{\delta} \frac{\Lambda^s(t^s)}{f^i(t^s)} \geqslant 0$ for all t^s, it is optimal[22] for each seller type t^s to set bidder-specific reserve bids, each of which is a price, $\tilde{t}_i$, satisfying

$$\tilde{t}^i - \frac{1-F^i(\tilde{t}^i)}{f^i(\tilde{t}^i)} = t^s + \frac{F^s(t^s) - \frac{1}{\delta}\Lambda^s(t^s)}{f^s(t^s)}.$$

Thus we can see that the *reserve bid principle* continues to hold. That is, the optimal auction corresponds to a direct mechanism in which the seller rejects any bid less than $\tilde{t}^i$. The standard

[21] For example, λ^s decreasing corresponds to a seller who is more concerned about earning profits when his valuation is low than when his valuation is high.

[22] We are still assuming the regular case, so this inequality will be satisfied as long as $\underline{t}^s \geqslant 0$.

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construction of the a^* indicates that this can be implemented by a second price auction with a reserve bid, where the second price is the maximum of $\widetilde{t}^i$ and the second highest bid. However, there are two important differences. First, the reserve bids must be made secret.[23] Second, the seller must commit to the (secret) reserve bid rule, since the reserve bid does not maximize interim expected profits, except in the special case of constant welfare weights when $F^s(t^s) = \Lambda^s(t^s)$. For example, if seller welfare weights are increasing then reserve bids will tend to be higher, since $F^s(t^s) > \Lambda^s(t^s)$, and the good is sold less often. If seller welfare weights are decreasing, then reserve bids will tend to be lower.

Finally, suppose the welfare weight on buyers is not zero. Then it becomes a generalization of the Myerson–Satterthwaite bargaining problem, with several buyers instead of just one buyer.

5.2.2. Assignment*: J objects, no sellers. Demand complementarities*

A related problem to auctions is an assignment problem where there are *N* buyers, $M < N$ objects, no seller, and each buyer demands at most one unit. In this case, for some distributions (uniform, for example) it is possible to achieve a first best efficient allocation. That is, the buyers with the *M* highest valuations are each allocated one unit [27].

The situation becomes more complicated and more interesting if the problem is to allocate *M* objects to *N* people who have preferences for bundles of discrete private goods. We will illustrate this with the special case of *single-minded buyers*, which is an extreme case of complementarity, and will look at ex ante efficient mechanisms with equal welfare weights, rather than revenue maximization.[24] We show in this section that unlike for the case of unit demands, the optimal allocation rules may not be efficient.

For each good *m* and each agent *i*, we denote their allocation by x^i_m, which equals 1 if object *m* is allocated to *i* and 0 otherwise. The payoffs are such that each person is identified by a unique subset $D^i \subseteq M$ of the *M* objects which they value as a bundle and a utility function such that $U^i = t^i q^i(x) - a^i$ where $q^i(x) = 1$ if $y^i_m = 1\ \forall m \in D^i$ and $q^i(x) = 0$ otherwise.[25] This is another special case of our structure in which interim efficient mechanisms reduce to picking *x* as follows:

$$\max_x \sum_{i=1}^{N} W^i(t^i) q^i(x)$$

$$\begin{aligned} \text{s.t.} \quad & q^i(x) \in \{0, 1\}, \\ & q^i(x) = \begin{cases} 1 & \text{if } y^i_m = 1\ \forall m \in D^i, \\ 0 & \text{otherwise,} \end{cases} \\ \text{and } & \textstyle\sum_{i=1}^{N} y^i_m = 1\ \forall m \in M \end{aligned}$$

and then minimize δ subject to IR. A full analysis of this problem is beyond the scope of this article, but a simple example offers some insight into the nature of efficient mechanisms in these environments.

[23] With publicly announced reserve bids, the best you can do is to set the interim profit maximizing reserve bid: $\widetilde{t}^i - \frac{1-F^i(\widetilde{t}^i)}{f^i(\widetilde{t}^i)} = t^s$.

[24] Levin [28] investigates a related model for the case of two goods.

[25] Ideally we would also like to analyze the case in which $u^i = \sum_{m \subseteq M} y^i_m t^i_m$. But that involves multi-dimensional types.

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Suppose $N = 3$, $S^1 = \{a\}$, $S^2 = \{b\}$, and $S^3 = \{a, b\}$. Further assume $t^i \sim$ uniform [0, 1] for each *i*. Therefore,

$$W^i(t^i, \delta) = t^i - \frac{\delta - 1}{\delta}\frac{1 - F}{f} = t^i - \frac{\delta - 1}{\delta}(1 - t^i).$$

The ex ante efficient mechanism awards the pair of items to 3 if $W^3(t^3, \delta) \geqslant W^2(t^2, \delta) + W^1(t^1, \delta)$. It awards them to 1 and 2 otherwise. In this case 3 wins iff $t^3 - \frac{\delta-1}{\delta}(1 - t^3) \geqslant t^1 - \frac{\delta-1}{\delta}(1 - t^1) + t^2 - \frac{\delta-1}{\delta}(1 - t^2)$ or iff $t^3 \geqslant t^1 + t^2 - \frac{\delta-1}{2\delta-1}$. So if individual rationality is binding, then $\delta > 1$ so both items will be awarded to 3 sometimes even though the first best allocation would award item *a* to 1 and *b* to 2. This inefficiency occurs for type realizations where $t^3 + \frac{\delta-1}{2\delta-1} > t^1 + t^2 > t^3$. The ex ante efficient auction is *not* first best efficient even in this case where types are one-dimensional and utility is linear in type.

If there is an option not to sell then *i* should be excluded when $t^i - \frac{\delta-1}{\delta}(1 - t^i) \leqslant 0$ or $t^i < \frac{\delta-1}{2\delta-1} \equiv \beta$. Then, if $t^i \geqslant \beta$ for all *i*, 3 wins iff $W^3 \geqslant W^1 + W^2$. 1 and 2 win otherwise. However, the goods will not be awarded to an agent for whom $t^i < \beta$. The full solution is summarized in the following table:

	$t^3 < \beta$	$t^3 > \beta$
$t^1 < \beta$, $t^2 < \beta$	No one	3 wins
$t^1 < \beta$, $t^2 > \beta$	Only 2 wins	3 wins if $t^3 \geqslant t^2$ 2 wins if $t^3 \geqslant t^3$
$t^1 > \beta$, $t^2 < \beta$	Only 1 wins	3 wins if $t^3 \geqslant t^1$ 1 wins if $t^1 \geqslant t^3$
$t^1 > \beta$, $t^2 > \beta$	1 and 2 win	3 wins if $t^3 + \beta \geqslant t^1 + t^2$ 1 and 2 win otherwise

For example, suppose $\delta = 3$. Then $\beta = .4$, buyer 3 is allocated both items with probability equal to .26, and buyers 1 and 2 each win their preferred item with probability equal to .51. Both goods are allocated only half the time, and sometimes the pair is allocated to buyer 3 when it is inefficient to do so. This outcome compares to a 100% allocation of both goods in the first best solution. With probability $\frac{1}{6}$ buyer 3 receives both items and with probability $\frac{5}{6}$ each of the other two buyers receive their preferred item. The main effect of the participation constraints in this example is that buyers 1 and 2 receive the item far less often than they should.

6. Conclusions

This paper presented a general framework to study the theoretical properties of interim efficient mechanisms in independent linear environments. Interim efficient allocation rules are fully characterized for these environments. For regular environments, the solution is often obtainable by applying classical welfare analysis, substituting easily computable *virtual utilities* for the agents' actual utilities. We illustrated this approach with a series of applications, some of which have been studied elsewhere in the literature, including both public goods and private goods applications. Other applications can also be analyzed in a similar way, including the problem of optimal cartel agreements [7], optimal reallocation of a jointly owned asset [6,11], optimal regulatory mechanisms [2], transfer pricing in organizations, and so forth.

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Several directions for future research seem promising. First, the incorporation of common or affiliated values can be done, at least for some specifications. For example, Myerson's [32] revision effects can be incorporated with only minor adjustments to the virtual valuations. A second issue, correlated types, involve some special features that we do not consider here, namely using complicated side-payments schemes that exploit the correlation in order to relax incentive constraints. These are used elsewhere, for example [10], and indeed can often relax incentive constraints fully, so that first best is achievable. However, due to the complicated nature of the sidepayments, these mechanisms may be impractical in most situations and also fail if there are limited liability constraints or if collusion is possible [21]. Third, there are interesting open questions about the asymptotic properties of interim efficient allocations. Fourth, the applications studied here concentrated on the regular case, and the exact details of efficient mechanisms for these applications in the irregular case is not fully solved.

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Appendix

Proof of Lemma 2. For each i and t, let

$$a^i(t) = \alpha^{0i} + \int_{\underline{t}^i}^{t^i} s\, dQ^i(s) - \frac{1}{N-1}\sum_{j\neq i}\int_{\underline{t}^j}^{t^j} s\, dQ^j(s)$$
$$+\frac{1}{N}\left[C(x(t)) - C^i(t^i) + \frac{1}{N-1}\sum_{j\neq i} C^j({}^j)\right],$$

where $C^i(t^i) = \int_T C(x(t))\, dF(t \mid t^i)$ and

$$\alpha^{0i} = A^{0i} + \frac{1}{N-1}\sum_{j\neq i}\int_{\underline{t}^j}^{\bar{t}^j}\int_{\underline{t}^j}^{t^j} s\, dQ^j(s)\, dF^j(t^j) - \frac{1}{N}\int_T C(x(t))\, dF(t).$$

If $a^i(t)$ is computed this way then for each t,

$$\sum_{i=1}^N a^i(t) = \sum_{i=1}^N \alpha^{0i} + C(x(t)).$$

Therefore, (x, a) is feasible if and only if $\sum_i \alpha^{0i} \geqslant 0$, or, equivalently,

$$\sum_{i=1}^N \left\{A^{0i} + \frac{1}{N-1}\sum_{j\neq i}\int_{\underline{t}^j}^{\bar{t}^j}\int_{\underline{t}^j}^{t^j} s\, dQ^j(s)\, dF^j(t^j) - \frac{1}{N}\int_T C(x(t))\, dF(t)\right\} \geqslant 0$$
$$\Leftrightarrow$$

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$$\sum_{i=1}^{N} A^{0i} + \sum_{i=1}^{N} \int_{\underline{t}^i}^{\bar{t}i} \int_{\underline{t}^i}^{t^i} s\, dQ^i(s)\, dF^i(t^i) - \int_T C(x(t))\, dF(t) \geqslant 0$$

$$\Leftrightarrow$$

$$S(x(\cdot)) \geqslant 0.$$

To verify that (x, a) is incentive compatible, observe first that $\frac{\partial Q^i}{\partial t^i} \geqslant 0$ by hypothesis and

$$\begin{aligned} A^i(t^i) &= \alpha^{0i} + \int_{\underline{t}^i}^{t^i} s\, dQ^i(s) \\ &\quad - \frac{1}{N-1} \sum_{j \neq i} \int_{\underline{t}^j}^{t^j} s\, dQ^j(s)\, dF^j(t^j) \\ &\quad + \frac{1}{N} \int_T C(x(t))\, dF(t) \\ &= A^{0i} + \int_{\underline{t}^i}^{t^i} s\, dQ^i(s), \end{aligned}$$

so both (IC1) and (IC2) are satisfied and $A^{0i} = A^i(\underline{t}^i)$ for all i.

Proof of Lemma 3 (*only if*). Let a be such that (x, a) is incentive compatible, feasible, and satisfies individual rationality. Incentive compatibility implies that there exist $\{A^{0i}\}_{i=1}^N$ such that

$$A^i(t^i) = A^{0i} + \int_{\underline{t}^i}^{t^i} s\, dQ^i(s) \quad \forall i, t^i.$$

The individual rationality constraint is

$$t^i Q^i(t^i) - A^i(t^i) - U^{0i}(t^i) \geqslant 0 \quad \forall i, t^i.$$

Combining the two gives:

$$t^i Q^i(t^i) - \int_{\underline{t}^i}^{t^i} s\, dQ^i(s) - A^{0i} - U^{0i}(t^i) \geqslant 0 \quad \forall i, t^i,$$

or, equivalently,

$$\min \left\{ t^i Q^i(t^i) - \int_{\underline{t}^i}^{t^i} s\, dQ^i(s) - U^{0i}(t^i) \right\} \geqslant A^{0i} \quad \forall i$$

or

$$\underline{t}^i Q^i(\underline{t}^i) + \min_{t^i} \left[\int_{\underline{t}^i}^{t^i} Q^i(s)\, ds - U^{0i}(t^i) \right] \geqslant A^{0i} \quad \forall i.$$

Summing over i gives:

$$\sum_{i=1}^{N} \underline{t}^i Q^i(\underline{t}^i) + \min_{t^i} \left[\int_{\underline{t}^i}^{t^i} Q^i(s)\, ds - U^{0i}(t^i) \right] \geqslant \sum_{i=1}^{N} A^{0i}.$$

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From resource feasibility and Lemma 3, $S(x(\cdot))\geqslant 0$, and hence

$$S(x(\cdot)) - \sum_{i=1}^{N} A^{0i} \geqslant -\sum_{i=1}^{N} A^{0i} \geqslant -\left\{\sum_{i=1}^{N} \underline{t}^i Q^i(\underline{t}^i) + \min_{t^i}\left[\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)\right]\right\},$$

which implies

$$\sum_{i=1}^{N}\left[\int_{\underline{t}^i}^{\bar{t}^i}\left(t^i - \frac{1-F^i(t^i)}{f^i(t^i)}\right) Q^i(t^i)\,dF^i(t^i) + \min_{t^i}\left[\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)\right]\right] - \int_T C(x(t))\,dF(t)\geqslant 0.$$

(if) For each i, let $A^{0i} = \underline{t}^i Q^i(\underline{t}^i) + \min_{t^i}[\int_{\underline{t}^i}^{t^i} Q^i(s)\,ds - U^{0i}(t^i)]$. Summing over i implies:

$$S(x(\cdot))\geqslant 0.$$

From Lemma 3, this implies the existence of a such that (x, a) is feasible and incentive compatible for all i and $A^{0i} = A^i(\underline{t}^i)$ for all i.

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